A THEORETICAL INVESTIGATION OF AN ARRESTED SALT WEDGE ESTUARY

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SUMMARY

The arrested salt wedge is a particular case of two layer flow. The equations of motion for this case contain interfacial, bed and wall shear terms. The shape and length of the intrusion of the arrested salt wedge can be determined theoretically from the governing equations if the laws of interfacial, bed and wall shear terms are known. No proper friction laws are available to evaluate these shear terms. In this paper, shear terms are evaluated by assuming the flow in the upper layer to be smooth turbulent and in the salt wedge to be laminar. A first order differential equation is obtained for defining the profile of the salt wedge. The validity of the assumptions is established by comparing the solution of the equation with available laboratory and field data.

1. INTRODUCTION

When the river flow is such that the volume inflow into the estuary during a tidal cycle is large compared to the tidal flow, the estuary will have a two layered flow separated by a thin layer of steep density gradient. This two layered estuarine flow consists of a protruding saline layer underneath the seaward flowing freshwater layer. In the absence of tides, the protruding sea water takes a stationary form in the shape of a wedge. This is termed as arrested salt wedge. The estuary in which the arrested salt wedge is formed, can be called an arrested salt wedge estuary. The arrested salt wedge flow patterns are a special case of two layer flow phenomenon. In the present investigation emphasis is laid only on this special case of two layer flow and a brief account of some of the earlier works on arrested salt wedge flows has been incorporated.

The shape and length of the intrusion of the arrested salt wedge can be determined theoretically from the governing equations if the laws of interfacial, bed and wall shear stresses are known. This paper presents the results of a theoretical investigation on the nature of shear parameters wherein suitable laws for interfacial, bed and wall shears are assumed and the solutions are compared with the experimental and field data. In order to accomplish this, the data of Farmer and Morgan (1), Keulegan (2) and Lesleighter (3) are used.

2. BASIC EQUATIONS

For an arrested salt wedge while the flow in the top portion of the sea water (lower) layer is in the direction of fresh water flow, the flow in lower portion of the sea water layer is in the opposite direction in accordance with the continuity equation. The flow pattern is given in Figures 1 and 2c. For a channel of uniform width B, the directions of shear stresses are as shown in Figures 2a and 2b and taken as positive, if acting in the direction of fresh water flow (x increasing direction). It may be noted here that the shear stress component in the flow direction is taken as the interfacial shear itself. This does not introduce any significant error except in the region very close to the toe of the wedge. Now referring to Figures 1 and 2, the integrated x momentum and continuity equations for the upper and lower layers can be written as follows:

x Momentum Equation

Upper Layer: $$B\rho_1 d_1 g \frac{d}{dx}(d_1 + d_2) + \rho_1 B \frac{d}{dx}\int_{d_2}^{d} |u_1|\, u_1\, dz = B\tau_i + 2d_1\tau \qquad \ldots (1)$$

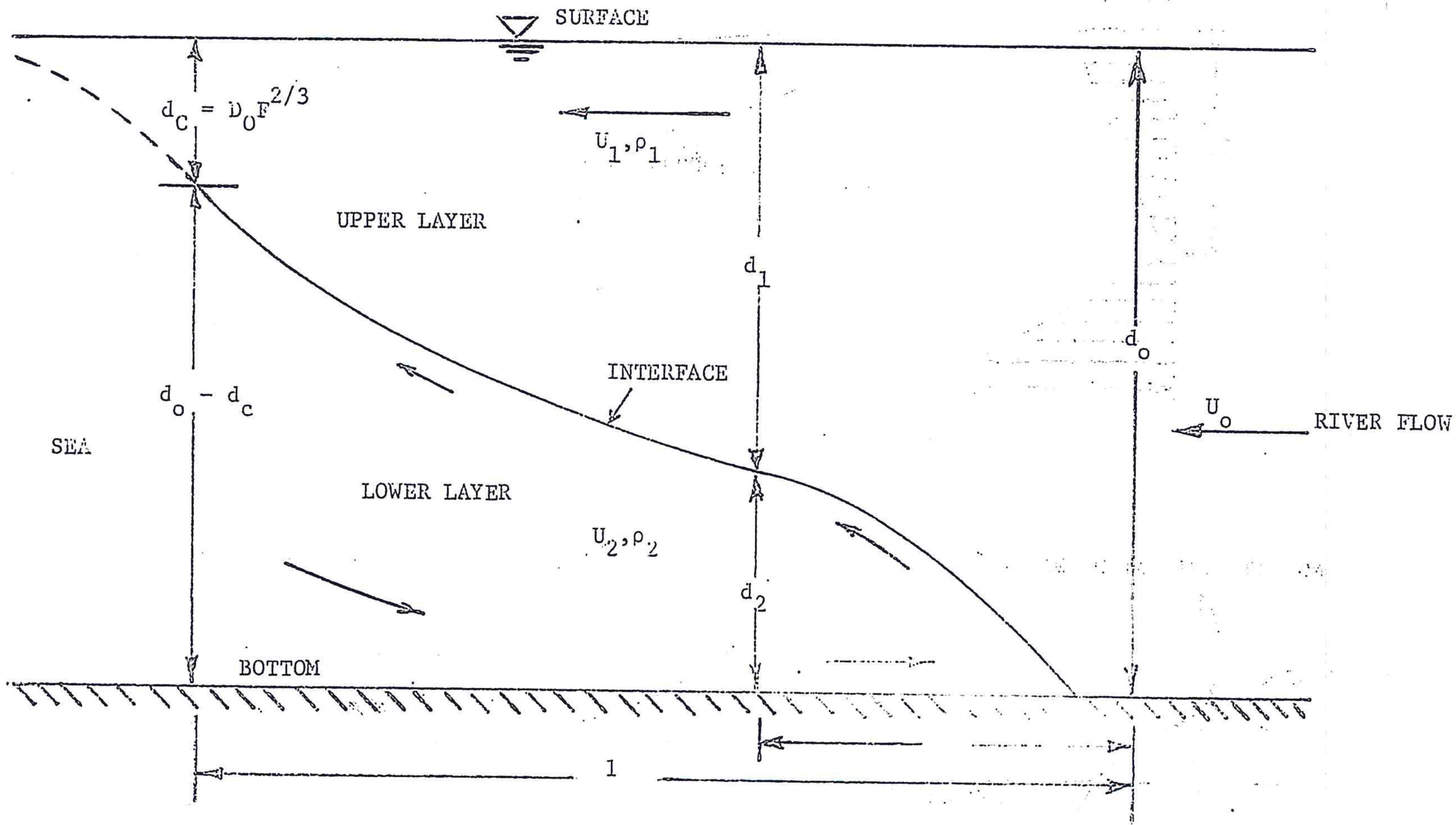

Fig. 1 - ARRESTED SALT WEDGE

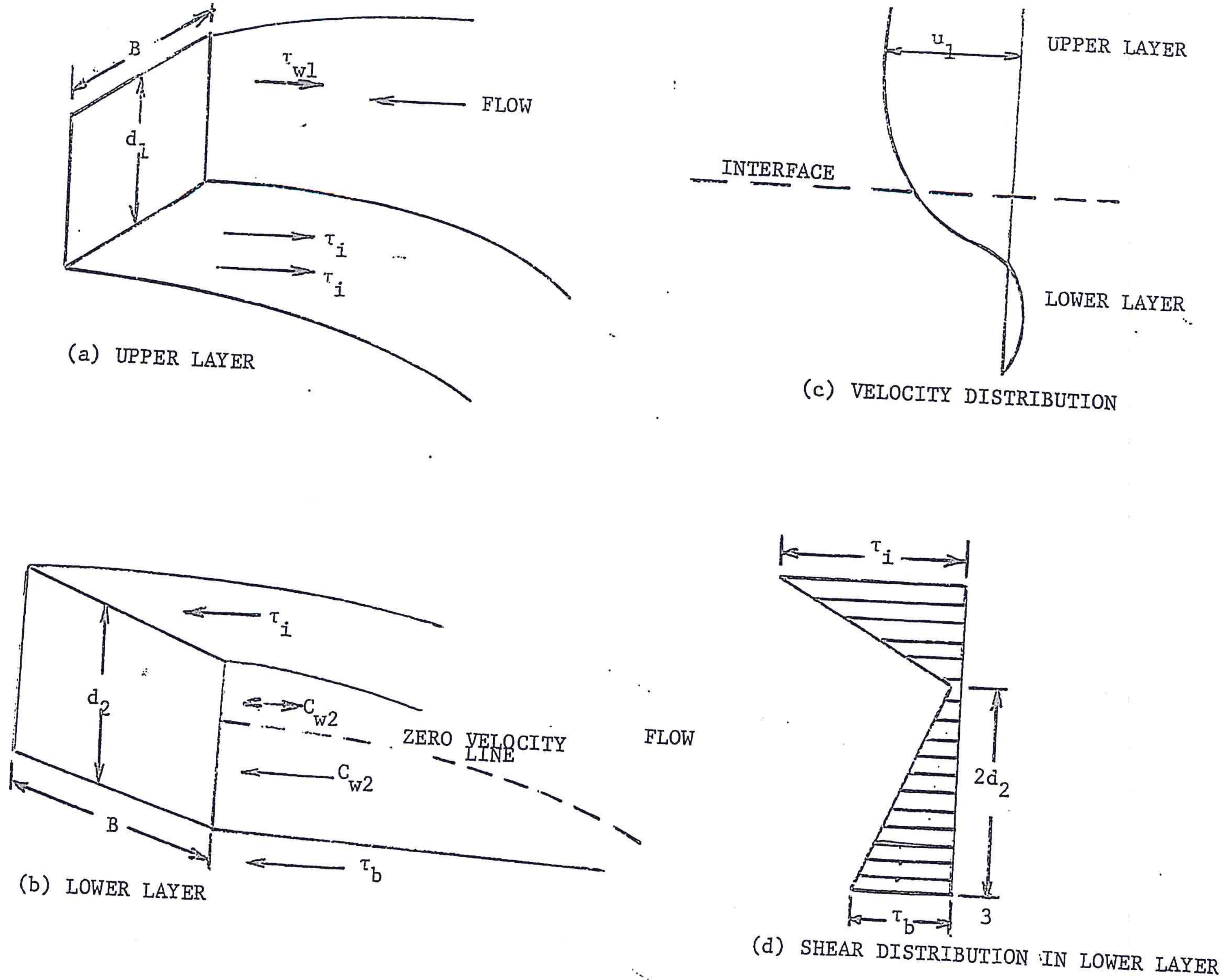

FIG. 2 - WALL, BOTTOM AND INTERFACIAL SHEAR STRESSES

Lower Layer: $Bd_2\, g \frac{d}{dx}\ (\rho_1 d_1 + \rho_2 d_2) +$

$$\rho_2\, B \frac{d}{dx} \int_o^d |\, u_2 \,|\, u_2\, dx = B\tau_i + B\tau_b + \int_o^{d_2} \tau_{w2}\, dz \qquad \ldots (2)$$

Continuity Equation

Upper Layer: $$B \int_{d_2}^{d} u_1\, d\, z = Q_1 \qquad \ldots (3)$$

Lower Layer: $$B \int_o^{d_2} u_2\, d\, z = o \qquad \ldots (4)$$

Where d is depth of the layer, g is acceleration due to gravity, Q is volume rate of flow, u is velocity in the flow direction, x and z are cartesian coordinates, x is the horizontal direction and is positive seawards and z is the vertical direction and is positive upwards, τ is shear stress, ρ is mass density and subscripts i, b and w refer to the interface, bed and wall respectively and subscripts 1 and 2 refer to the upper and lower layers respectively and subscript o refers to conditions that x = o, that is at the toe of the wedge.

The following assumptions are made to obtain Equations 1 to 4:

(i) All the variations in the direction of the width of the estuary are negligible.

(ii) A sharp interface exists which is taken as that streamline separating fresh water and salt water coincident with density discontinuity.

(iii) The free surface shear is zero.

(iv) Entrainment of the salt water into the upper layer is small and the density ρ_1 remains constant along the length of the wedge.

(v) The net flow is zero across any section in the lower layer.

In the earlier investigations of Farmer and Morgan (1) and Schijf and Schoenfeld (4), τ_b, the bed shear stress is assumed to be zero. Actually the magnitude of τ_b is comparable to that of τ_i and hence τ_b

is not to be neglected in Equation 2. The velocity profiles obtained by Keulegan (2) and Lesleighter (3) within the salt wedge can be described fairly well by the following laminar flow equation as shown in Figure 3. This is discussed in detail by Muralikrishna (5).

$$\frac{u_2}{u_i} = 3\left(\frac{z}{d_2}\right)^2 - 2\left(\frac{z}{d_2}\right) \qquad \ldots (5)$$

From this it can be shown that

$$\tau_i = \left(\frac{d\,u_2}{dz}\right)_{\text{at } z = d_2} = 2\left(\frac{d\,u_2}{dz}\right)_{\text{at } z = 0} = 2\tau_b \qquad \ldots (6)$$

As pointed out by Keulegan (2), the term $\int_0^{d_2} \tau_{w2}\, dz$ in Equation 2 can be neglected since the net effect of the shear force in the lower layer resulting from the velocity reversing in direction along the depth is small (Figure 2).

Two more assumptions are made at this stage viz.

(vi) The velocity u_1 in the upper layer is uniform along the depth (Figure 2a).

(vii) $\frac{d}{dx}(d_1 + d_2) = 0.$

These two assumptions are the same as the assumptions made by Schijf and Schoenfeld (4).

Using the Keulegan's experimental result $u_i = 0.63\, u_1$ and combining it with Equation 5, we get:

$$\frac{d}{dx}\int_0^{d_2} |\, u_2 \,|\, u_2\, dz = 0.058\, \frac{d}{dx}(u_i^2\, d_2)$$

$$= 0.021\, \frac{d}{dx}(u_1^2\, d_2) \qquad \ldots (7)$$

The simplified equations are:

$$\frac{1}{gd_1}\frac{d}{dx}(u_1^2\, d_2) = -\frac{\tau_i + 2\,\tau_{w1} A}{\rho_1 g d} \qquad \ldots (8)$$

$$\text{and } -J\frac{d\,d_1}{dx} + \frac{0.021}{g\,d_2}\frac{d}{dx}(u_1^2\, d_2) = \frac{3\,\tau_i}{2\,\rho_2 g d_2} \qquad \ldots (9)$$

where $A = d_1/B$ and $J = (\rho_2 - \rho_1)/\rho_2$

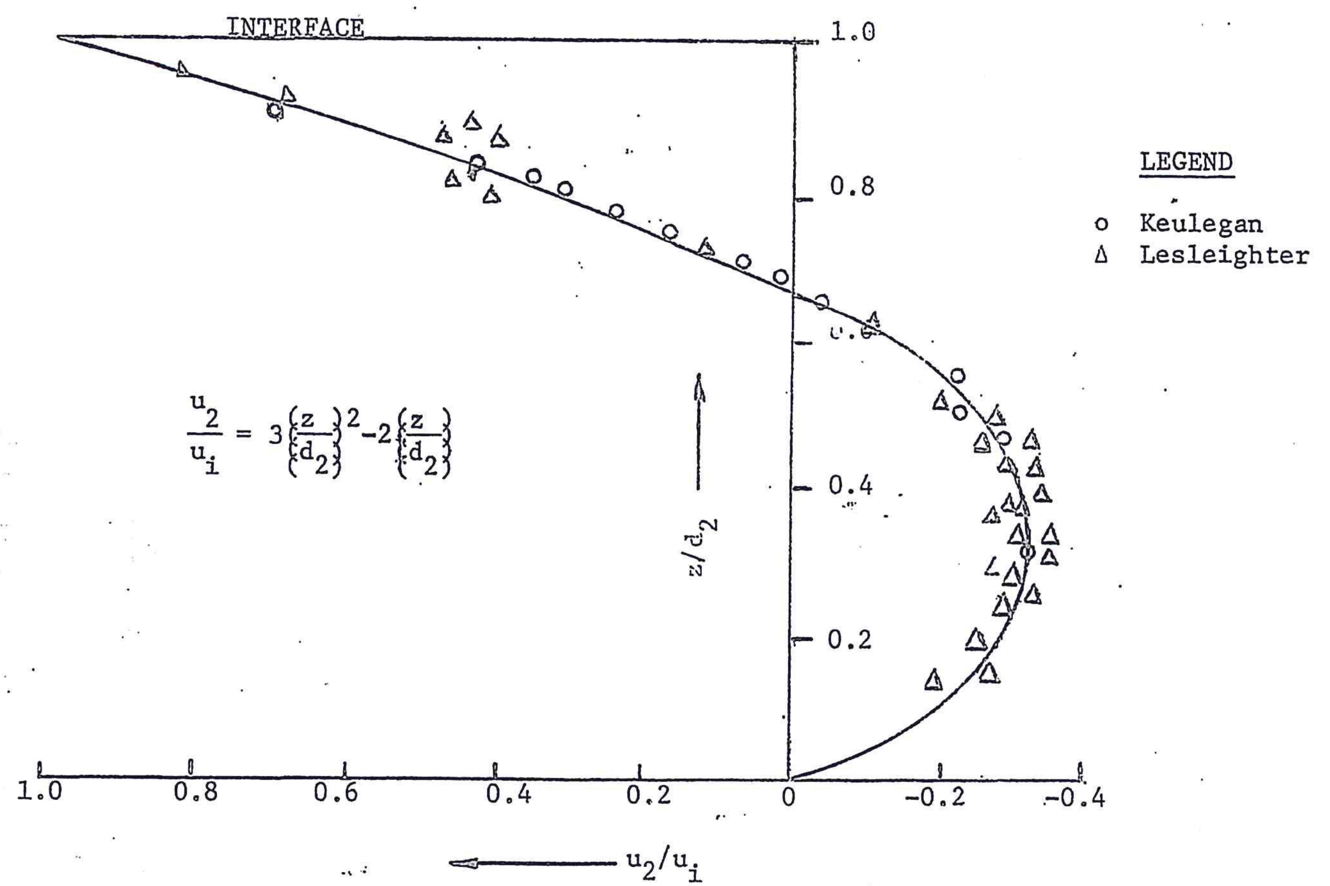

FIG. 3 - VELOCITY DISTRIBUTION IN LOWER LAYER

3. LAW OF FRICTION FOR INTERFACE AND SIDE WALLS

Equations 8 and 9 can be integrated if τ_i and τ_w are known in terms of the flow parameters and fluid properties of the upper layer. No clear conclusion concerning interfacial and wall shear stresses for the case of arrested salt wedge can be made from the literature known to the authors.

With the definition of the Reynolds number as $Re = u_o d_o / \nu$ it may be expected (6) for laminar flow that K(Re) while for turbulent flow the influence of the Reynolds number may be insignificant so that $K = f(F)$ only. Here F is densimetric Froude number defined as $F = U_o/\sqrt{gJ\, d_o}$. Vreugdenhil (6) in his attempt to draw a workable conclusion concerning interfacial shear for the general case of two layer flows suggests that it is not possible to define a relationship between K, Re and F. For the case of an arrested salt wedge, Vreugdenhil (6) however concludes that a relationship exists between K and Re and probably there is also a dependence on F for large values of Re.

Farmer and Morgan (1) assumed in their analysis of very wide estuaries,

$$\tau_i = K \rho_2 u_o^2 \qquad \ldots (10)$$

Where u_o is the value of u_1 at $x = o$ and K is a friction coefficient. In their analysis K is not given in terms of the flow parameters and estuary geometry but is to be evaluated separately for every experimental set-up or estuary. Also K is believed to remain reasonably constant for a given set-up or estuary.

Keulegan, on the basis of his experiments has proposed the following laws for interfacial and bed shear stresses:

$$\frac{\tau_i}{\rho_1 u_o^2} = 17.5 \left(\frac{u_o d_o}{\nu} \right)^{-1} \qquad \ldots (11)$$

and

$$\frac{\tau_b}{\rho_2 u_o^2} = 0.031 \left(\frac{u_o d_0}{\nu} \right)^{-1/4} \qquad \ldots (12)$$

where $d_o = d_1$ at $x = o$ and is equal to $d_1 + d_2$ for other values of x and ν is the kinematic viscosity. It may be noted here that the procedure followed by Keulegan to obtain τ_i amounts to evaluating τ_{wl} by ignoring the presence of the interface. τ_{wl} is evaluated by treating the flow in the upper layer as two dimensional smooth turbulent flow between parallel walls separated by a distance equal to B. Then, using this value of τ_{wl} in the x-momentum equation for upper layer Keulegan obtains τ_i. Evaluating τ_i thus, he proceeds to evaluate τ_b from the x-momentum equation for the lower layer. The shear stresses so evaluated are expressed in the form of Equations 11 and 12. These equations suggest that the interfacial shear stress is laminar and the bottom one turbulent. These results are strange even according to Keulegan. Partheniades, Dermiss and Mehta (7) point out that the law proposed for interfacial shear predicts shear values which are smaller by two or three orders of magnitude than the actual values for the prototypes. This discrepancy appears to be due to the procedure adopted by Keulegan to obtain τ_i.

From this discussion it appears that τ_i is a function of the upper layer Reynolds number but the law of friction is not laminar. So it would be logical to assume that the upper layer is smooth turbulent. This implies that the sides and interface behave as smooth walls. So τ_i and τ_w are replaced by τ_a which is an average shear term. The authors assume at this stage that the Blasius law of friction is valid. The following equations can now be written:

$$\tau_{wl} = \tau_i = \tau_a = \frac{f}{8} \rho_1 u_1^2 \qquad \ldots (13)$$

and

$$f = \frac{0.316}{(4u_1 R_1/\nu)^{0.25}} \qquad \ldots (14)$$

where R_1 is the hydraulic radius $= \dfrac{B\, d_1}{B + 2d_1}$.

As the customary practice of replacing pipe diameter by 4 R in defining Reynolds number for the cases of open channel flows is found to give satisfactory results, Blassius' law is written in the form of Equation 14, which is expected to be applicable in the range where 1/7th power law distribution is valid.

4. DIFFERENTIAL EQUATION FOR THE PROFILE OF THE ARRESTED SALT WEDGE

The differential equation for the profile of the arrested salt wedge is given in terms of the following non-dimensional parameters:

$$D = d_1/d_o, \qquad X = x/d_o,$$

$$A = do/B, \qquad U = u_1/u_o \qquad \ldots (15)$$

and R_Δ the densimetric Reynolds number $= \dfrac{gJ\, d_o^{3/2}}{\nu}$.

R_Δ is actually the ratio of a Reynolds number and a densimetric Froude number.

Subtracting Eq. 8 from Eq. 9 and substituting Equations 13, 14 and 15, noting $d_2 = d_o = d_1$ and $UD = 1$ the following differential equation is obtained for the profile of the arrested salt wedge:

$$\frac{dD}{dX} = \frac{0.0279\ F^2\ (1 + 2AD)^{0.25}}{(R_\Delta F)^{0.25}}\ \frac{(1+2AD)\ (1-D)+1.5\ D(1-J)}{F^2(1-D)-D^3\ (1-D)-0.021F^2(2-D)} \quad \ldots\ (16)$$

The profile of the arrested salt wedge is obtained by solving Equation 16 by numerical integration by Runge-Kutta method for a given set of F, R_Δ, A and J values. Farmer and Morgan (1) observed that the densimetric Froude number of the upper layer :

$$F_I \left(= \frac{u_1}{\sqrt{gJd_1}}\right)$$

approaches unity at the river mouth. This condition gives, $d_c = d_o\ F^{2/3}$, where d_c is the critical depth of the upper layer at the river mouth. This condition is made use of in obtaining l, the length of the wedge (Figure 1).

5. EXPERIMENTAL DATA

The differential equation so obtained is verified with the experimental data of Farmer and Morgan (1), Keulegan (2) and Lesleighter (3) The field data of Mississippi River South West pass as given by Farmer and Morgan (1) is also used for establishing the validity of the assumed shear law for the limiting case of A = 0, The details of the data used are given in Table 1. The data corresponding to values of J less than 0.03 is chosen for comparison as, usually the upper bound value for J is around 0.03 for the case of the salt wedge.

TABLE 1

EXPERIMENTAL DATA

Investigator	R_Δ	F	A	J
Keulegan (2)	13300	0.15 to 0.30	2.11	0.011
	17300	0.15 to 0.30	2.11	0.020
	22657	0.19 to 0.40	2.11	0.0046
Lesleighter (3)	32300	0.37 to 0.60	0.66	0.028
Farmer and Morgan (1)	-	-	1.0	0.000684 to 0.025
Field data: Mississippi river South West Pass	R_Δ = 24,774,245 J = 0.020 d_o = 1463 cm, A = 0.03 F = 0.25			From Farmer and Morgan (1)

6. LENGTH AND SHAPE OF THE WEDGE

A log-log plot of the lengths of the wedge, obtained in the present study with F (Figures 4 and 5) reveals a satisfactory agreement with the experimental data of Keulegan and Lesleighter. Lesleighter conducted experiments for flows with F greater than 0.03, and his equation for length of the wedge (R_Δ = 32300, J = 0.028, A = 0.66) is

$$L = \text{constant } (F^{-4}) \qquad \ldots (17)$$

where $L = 1/d_o$. The range of densimetric Froude Numbers for which the experiments were conducted by Keulegan (for J < 0.03) is 0.15 to 0.40. For this range of F, his empirical ralation for the length of the wedge is in the form,

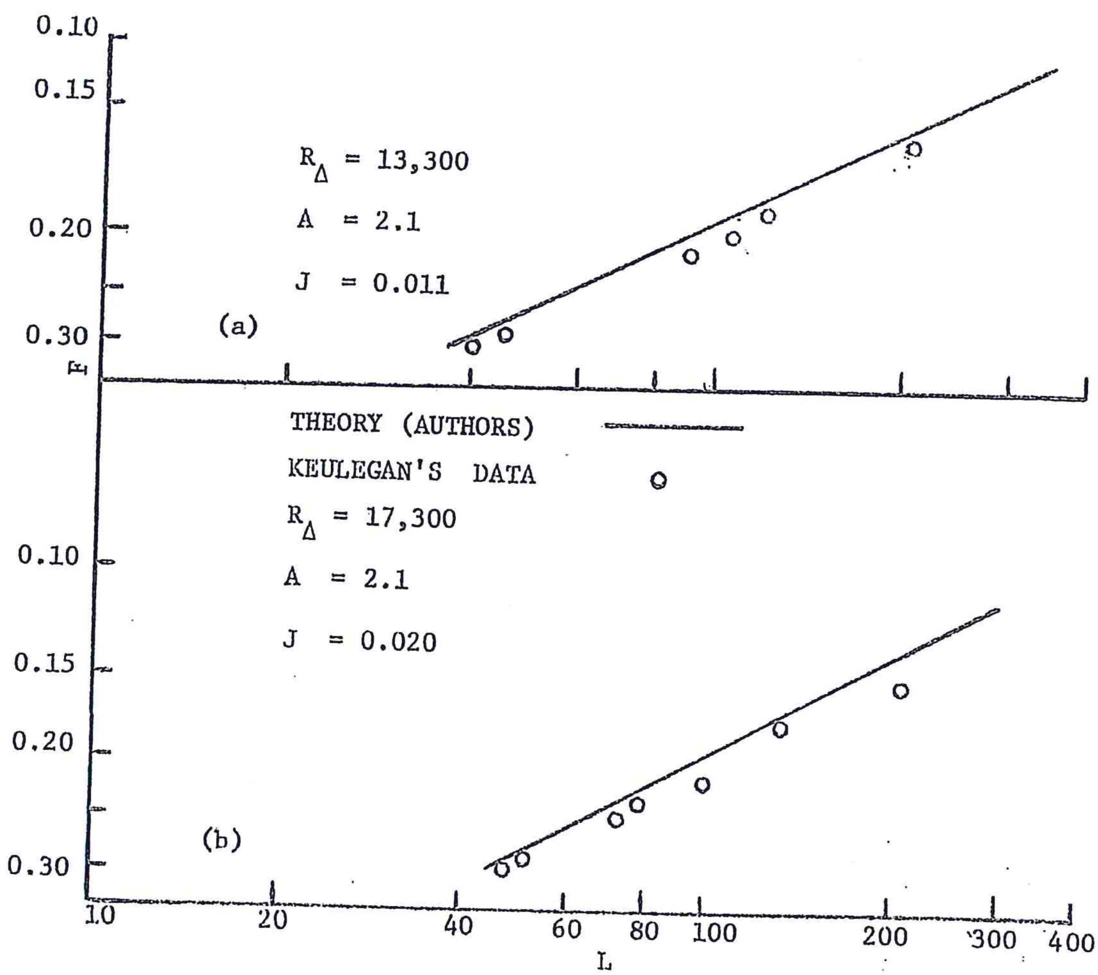

FIG. 4 - LENGTH OF ARRESTED SALT WEDGE

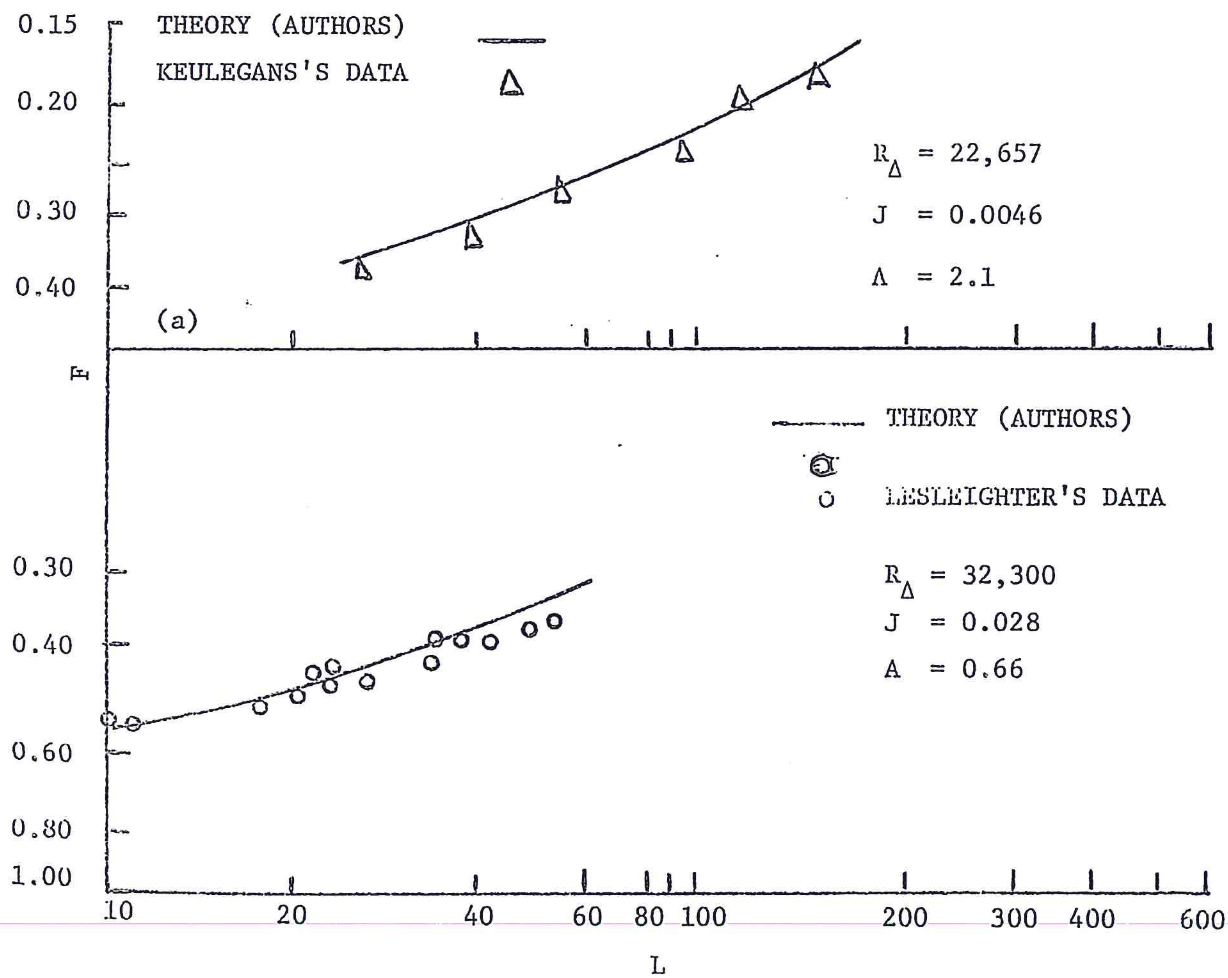

FIG. 5 - LENGTH OF ARRESTED SALT WEDGE

$$L = \text{constant } (F^{-5/2}) \quad \ldots (18)$$

From Figures 4 and 5 it can be seen that authors theoretical solution gives a similar relationship. The law for length of the wedge can be written in the following general form:

$$L = C\ (F)^{-m} \quad \ldots (19)$$

A comparison of the values of the co-efficient C and power m as obtained by Keulegan, Lesleighter and authors is given in Table 2. In three cases

TABLE 2

COMPARISON OF EXPERIMENTALLY AND THEORETICALLY EVALUATED VALUES OF m AND C

R_Δ	J	A	F	From Keulegan's equation		From Author's Theory		From Lesleighter's equation	
				m	C	m	C	m	C
13300	0.011	2.11	0.15 to 0.30	2.5	2.30	2.25	2.5	-	-
17300	0.020	2.11	0.15 to 0.30	2.5	2.7	2.00	3.7	-	-
22657	0.0046	2.11	0.19 to 0.40	2.5	3.19	2.80	3.12	-	-
32300	0.028	0.66	0.37 to 0.60	-	-	4.00	1.06	4.0	1.14

out of four the agreement can be seen to be good. For the case of J = 0.02 the approximate theoretical curve of the author and the empirical curve proposed by Keulegan do not seem to agree well but it can be seen from Figure 4 that there is not much difference in values of lengths computed from the authors model and Keulegan's measured values for this case.

For the data of Mississippi river given in Table 1, Farmer and Morgan evaluated the friction term taking 1 the length of the wedge as 16.67 kilometers. The present theoretical analysis for the data given in Table 1 gives a value of 11.3 kilometers for the length of the wedge which is about 33% smaller than the measured value. Errors of this magnitude are quite acceptable when comparing theoretical solutions with the field data. The error can be explained as follows. Let us assume that there is some error in the measured value of F. Let the actual value be 0.23 and not 0.25 as given. This could result either due to a 7% error in the measurement of the depth d_o or 8% error in the measurement of discharge per unit width. Such a small change in F would bring about a change of 20% in the length of the wedge. This argument is advanced only to show how sensitive the length of the wedge is to a small error in F.

Figures 6 to 10 show the non-dimensional shapes of the arrested salt wedge. According to Keulegan the shape of the wedge is a function of X/L and independent of F, A and J. The plot of the shape of the wedge (Figure 6a) based on the present theoretical analysis shows considerable effect of the densimetric Froude number F. This has been pointed out by Partheniades et al also. Figure 6b gives the shape of the wedge for F = 0.2. It indicates no noticeable effect of J on the shape of the wedge. This is in agreement with Hinwood's (8) finding. The agreement between the authors theoretical solution and experimental data of Keulegan for the shape of the wedge (Figures 7 a,b and Figures 8 a,b) is fairly good with the exception of the case for F = 0.294 (Figure 7a). The theoretical shape of the wedge is shown along with the mean experimental curve of Lesleighter in Figure 9. The agreement is good. A comparison of the shape of the wedge with field data of South West pass, Mississippi river and experimental data of Farmer and Morgan shown in Figure 10. The agreement is reasonably good. Considering the agreement between the present model and the observed data as a whole (Figures 7 to 10) it is concluded that the profile of the wedge is well described by the differential equation 16.

7. FRICTION LAW

On the basis of the agreement between the measured lengths of the salt wedge and the predicted ones which is shown in Figures 4 and 5, it is concluded that the validity of the friction law assumed is established.

However, one could argue that the validity of the assumptions has been established only for large values of A (>0.65) where the contribution of interfacial shear is not a substantial part of the total shear. So, if the model could be verified for very small values of A where the contribution of interfacial shear is a very substantial part of the total shear, then the validity of the assumptions would be established beyond question. In this context, it may be mentioned that the data available to the author for small values of A is limited to South West pass, Mississippi river. In this case the predicted value of length of the wedge is short by about 33%. This disagreement is not very much and as mentioned previously could be due to errors which usually are associated with measurementsoof field data. It is hoped that some more field data would be available in future to establish further, the validity of the model.

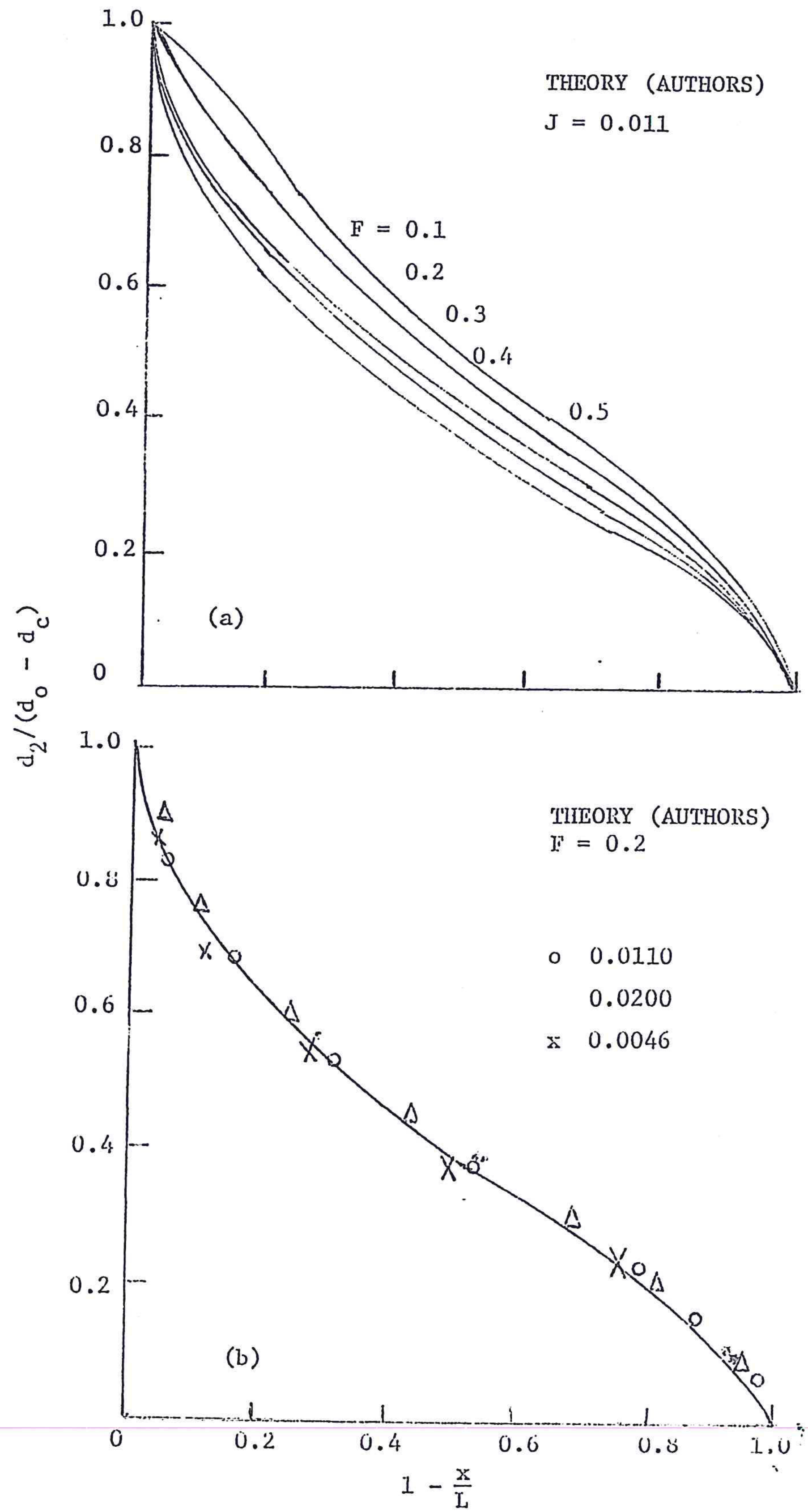

FIG. 6 - SHAPE OF ARRESTED SALT WEDGE

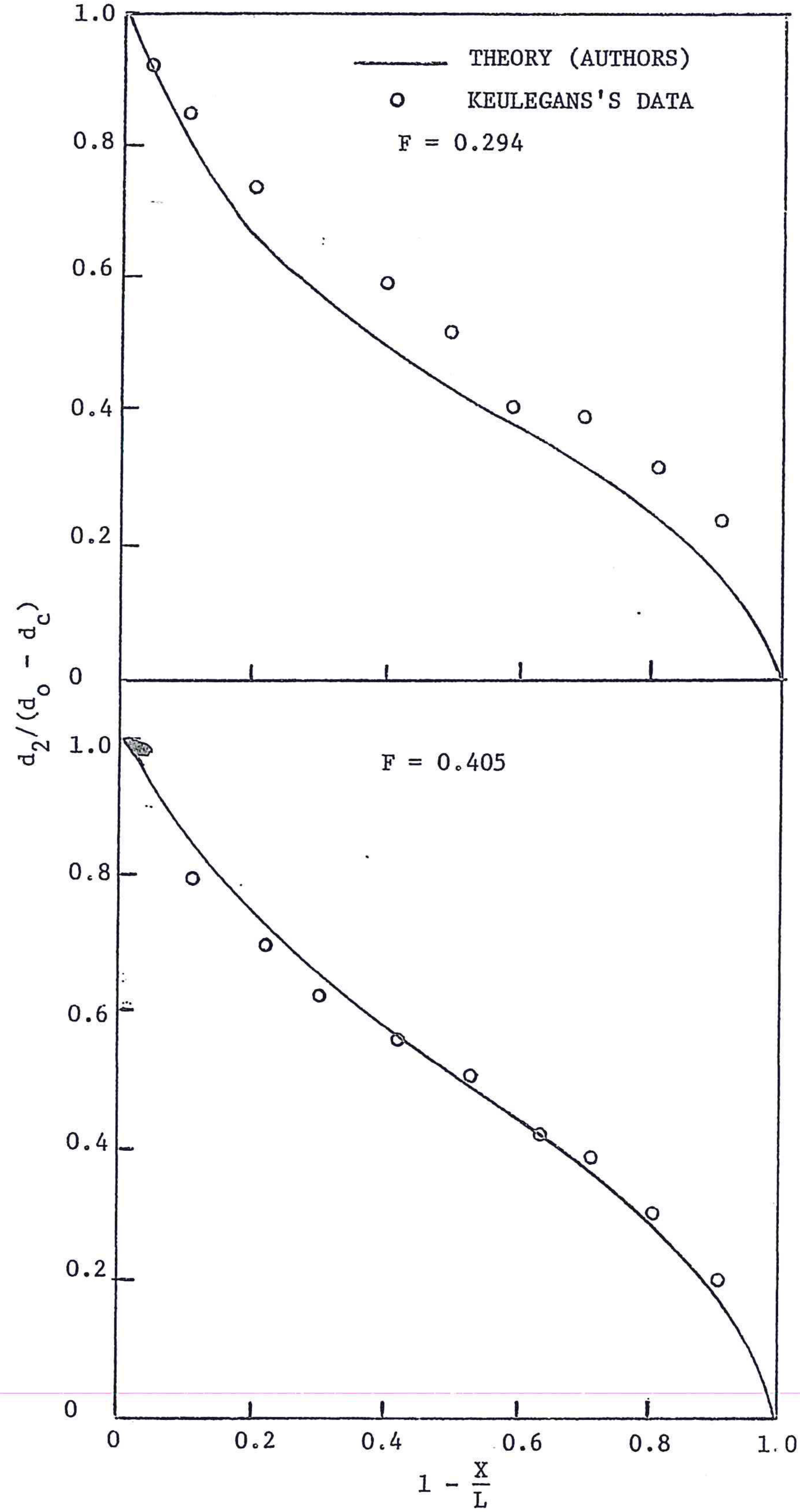

FIG. 7 - COMPARISON OF SHAPE OF ARRESTED SALT WEDGE

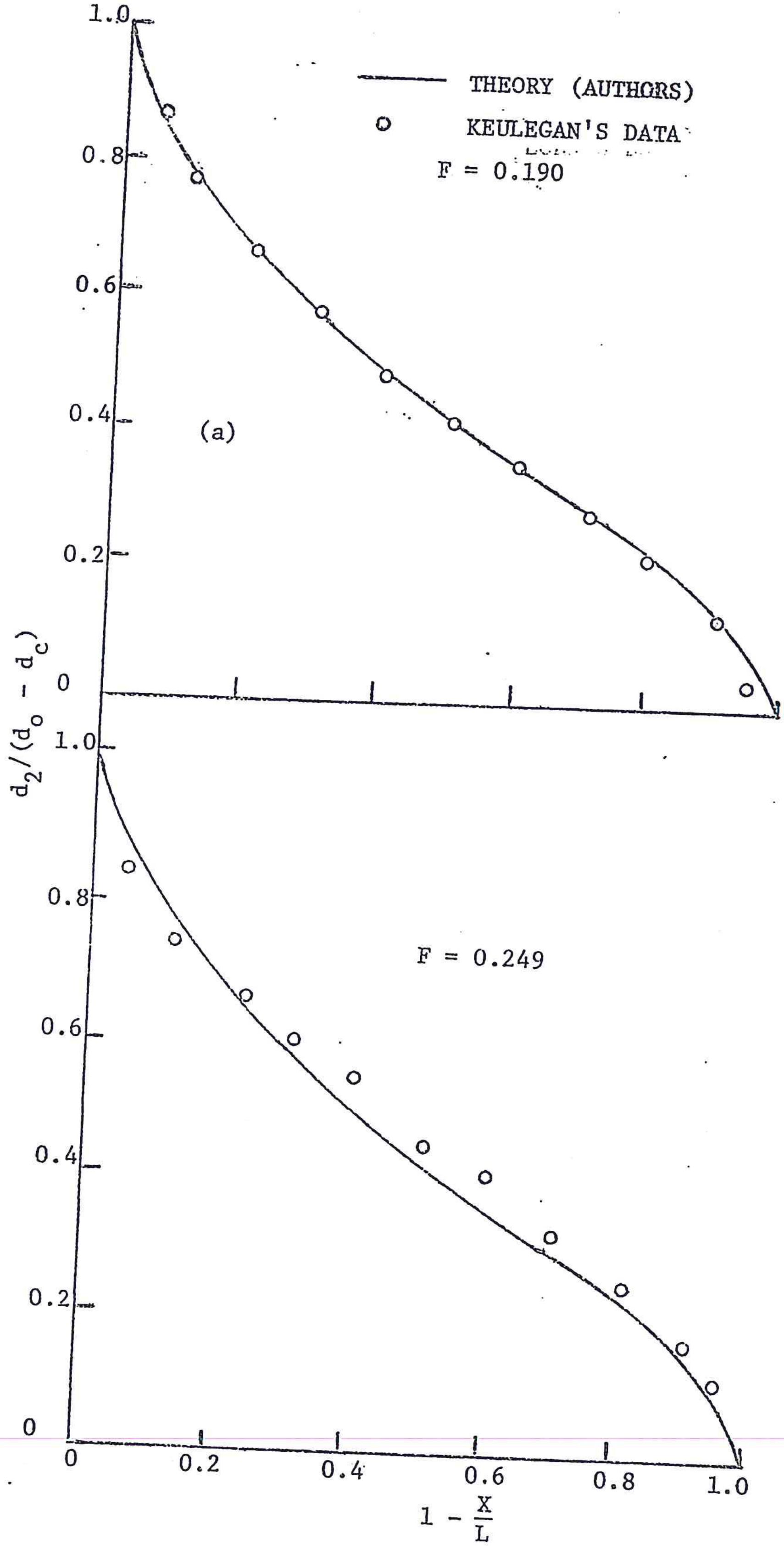

FIG. 8 - COMPARISON OF SHAPE OF ARRESTED SALT WEDGE

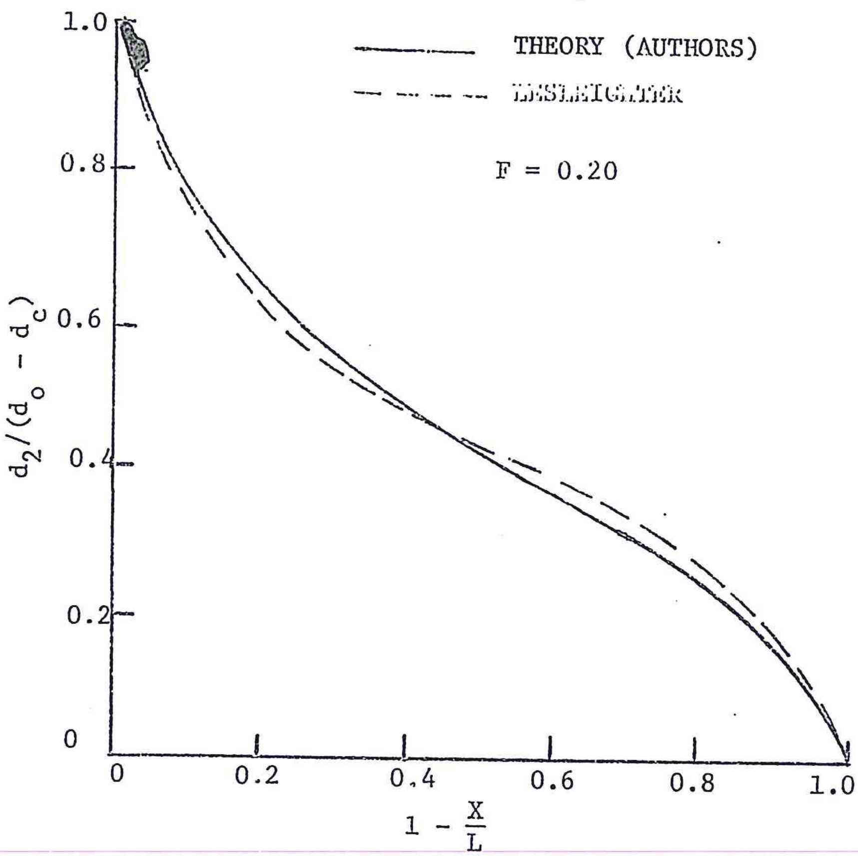

FIG. 9 - COMPARISON OF SHAPE OF ARRESTED SALT WEDGE

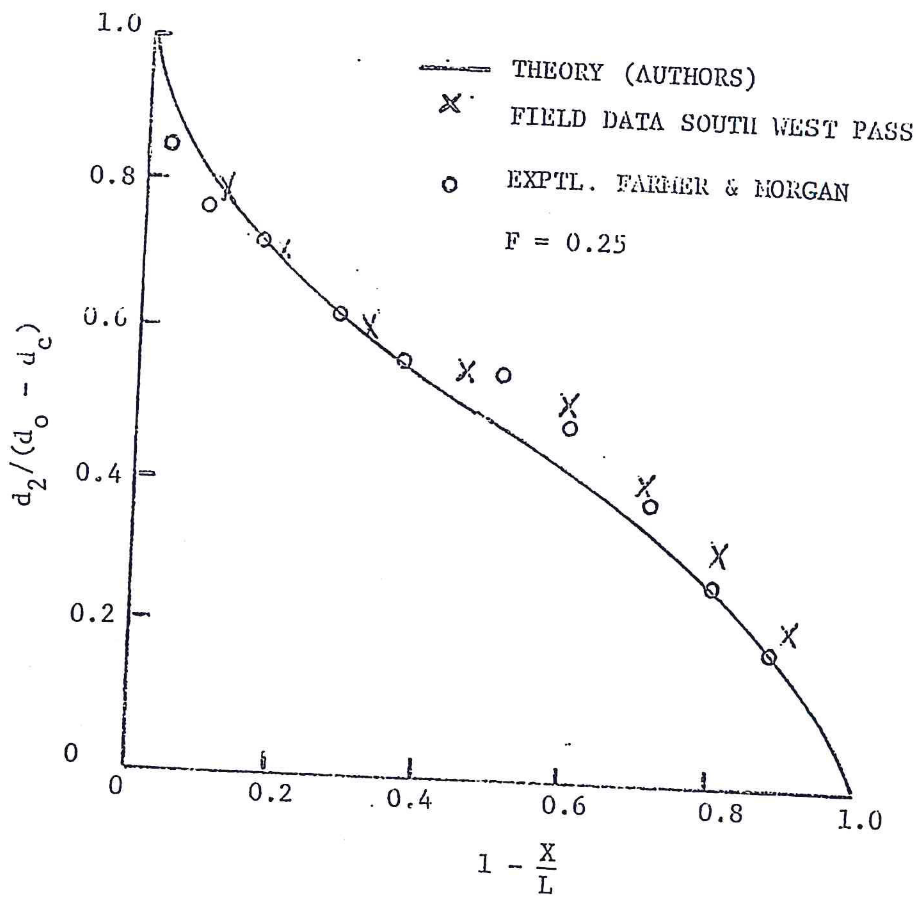

FIG. 10 - COMPARISON OF SHAPE OF ARRESTED SALT WEDGE

8. CONCLUSION

The theoretical solution proposed in this paper based on the assumptions that flow in the upper layer is turbulent and smooth and the friction is given by Blasius' law, gives results that agree well with experimental findings. The assumptions of laminar flow in the lower layer which gives $\tau_b = \tau_i/2$ is reasonable. The profile and length of the arrested salt wedge can be determined by using Equation 16. While the shape of the arrested salt wedge depends significantly on F, it does not seem to depend on J.

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